stichting
mathematisch
centrum

AFDELING MATHEMATISCHE STATISTIEK SW 20/73 OCTOBER

Y. LEPAGE

ASYMPTOTICALLY OPTIMUM RANK TESTS FOR CONTIGUOUS LOCATION AND SCALE ALTERNATIVES

Prepublication

2e boerhaavestraat 49 amsterdam

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970) : 62G10, 62G30

Contents

Asymptotically optimum rank tests for contiguous location and scale alternatives *)

Yves Lepage **)

Abstract

The problem of testing identity of distribution against alternatives containing both location and scale parameters is studied. Conditions are given to obtain contiguous location and scale alternatives and, for those alternatives, an asymptotically most powerful rank test is found. The results are then specialised to the two-sample case.

1. Introduction

In the paper of Hájek (1962) and the book of Hájek and Šidák (1967), the problem of testing the null hypothesis of randomness versus contiguous location alternatives or contiguous scale alternatives was treated. In each case, an asymptotically most powerful rank test is found. In this paper, the problem of testing the null hypothesis of randomness versus contiguous location and scale alternatives is considered. The approach adopted follows that of Hájek and Šidák (1967) and many of our proofs are similar to theirs.

Section 2 contains the basic notations and tools that will be needed. In section 3, conditions are given to provide contiguous location and scale alternatives and the asymptotic distribution of linear rank statistics under such contiguous alternatives is found. In section 4, the notion of asymptotic sufficiency is explored to deduce a rank test asymptotically most powerful among all tests while in section 5 all the results are specialised to the two-sample case. Sections 6, 7 and 8 contain the proof of the results of respectively sections 3, 4 and 5.

---

*) This work is part of the author's Ph.D. dissertation written at the Université de Montréal under the direction of Professor Constance van Eeden and, it was partially supported by the National Research Council of Canada, Grant No. A-8555. The manuscript was completed while the author was visiting the Mathematisch Centrum, Amsterdam. The paper is not for review; it has been submitted for publication in a journal.

**) Université de Montréal; temporarily: Mathematisch Centrum.

2. Notations and conditions

Let $N_\nu (\nu=1,2,\ldots)$ be a sequence of positive integers such that $N_\nu \to \infty$ when $\nu \to \infty$. For each ν, consider a sequence of random variables $X_{\nu 1},\ldots, X_{\nu N_\nu}$ and denote by $R_{\nu i}$ the rank of $X_{\nu i}$ among $X_{\nu 1},\ldots, X_{\nu N_\nu}$.

Suppose that under H_ν, the random variables $X_{\nu 1},\ldots, X_{\nu N_\nu}$ are independently and identically distributed according to a continuous distribution and that under K_ν, the joint density of $(X_{\nu 1},\ldots,X_{\nu N_\nu})$ is given by

$$(2.1) \qquad q_\nu = \prod_{i=1}^{N_\nu} e^{-c_{\nu i}} f(e^{-c_{\nu i}} x_i - d_{\nu i})$$

with $c_\nu = (c_{\nu 1},\ldots,c_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$, $d_\nu = (d_{\nu 1},\ldots,d_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$ and a known density f in the class C of absolutely continuous density functions on $\mathbb{R}$ such that

$$(2.2) \qquad I(f) = \int_0^1 \phi^2(u,f)du < \infty, \; I_1(f) = \int_0^1 \phi_1^2(u,f)du < \infty$$

and

$$(2.3) \qquad \int_0^1 \phi(u,f)du = \int_0^1 \phi_1(u,f)du = 0$$

where if F(x) is the distribution function corresponding to f(x),

$$(2.4) \qquad \phi(u,f) = -\frac{f'(F^{-1}(u))}{f(F^{-1}(u))} \text{ and } \phi_1(u,f) = -1 - F^{-1}(u)\frac{f'(F^{-1}(u))}{f(F^{-1}(u))},$$

$0 < u < 1$.

Let $\bar{c}_\nu = \sum_{i=1}^{N_\nu} c_{\nu i}/N_\nu$, $\bar{d}_\nu = \sum_{i=1}^{N_\nu} d_{\nu i}/N_\nu$, $c_\nu^0 = (c_{\nu 1}-\bar{c}_\nu,\ldots,c_{\nu N_\nu}-\bar{c}_\nu)$ and $d_\nu^0 = (d_{\nu 1}-\bar{d}_\nu,\ldots,d_{\nu N_\nu}-\bar{d}_\nu)$. We now define some sets of conditions for the vectors c_ν and d_ν.

Condition A.

(i) $\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 = 0.$

(ii) For $\nu = 1,2,\ldots,\ c_{\nu i} - \bar{c}_\nu \neq 0\ (i=1,\ldots,N_\nu)$.

(iii) There exists a real number K such that
$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (e_{\nu i}(c_{\nu i}-\bar{c}_\nu)^{-1}-K)^2 = 0$ where

$e_{\nu i} = d_{\nu i} - \bar{d}_\nu.\exp(-c_{\nu i}+\bar{c}_\nu),\ i = 1,\ldots,\ N_\nu.$

It is easily seen that condition A implies $\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} e_{\nu i}^2 = 0$.

For $K \in \mathbb{R}$ and $f \in C$, define

$$(2.5)\qquad I(f,K) = \int_0^1 \phi^2(u,f,K)du$$

where

$$(2.6)\qquad \phi(u,f,K) = K\phi(u,f) + \phi_1(u,f),\ 0 < u < 1.$$

Condition B.

(i) Condition A is satisfied.

(ii) For $f \in C$, $\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \ .\ I(f,K) = b^2$ where $0 < b^2 < \infty$.

Consider a sequence of subsets M_ν of $\mathbb{R}^{N_\nu} \times \mathbb{R}^{N_\nu}$. We will define for the vectors $(c_\nu,d_\nu) \in M_\nu$, an analogue of conditions A and B by the following statement.

Condition M.

(i) $\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu) \in M_\nu} \max_{1\le i\le N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 = 0.$

(ii) For each $(c_\nu,d_\nu) \in M_\nu$, $c_{\nu i} - \bar{c}_\nu \neq 0\ (i=1,\ldots,N_\nu;\ \nu=1,2,\ldots)$.

(iii) There exists a real number K such that

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \max_{1\le i\le N_\nu} (e_{\nu i}(c_{\nu i}-\bar{c}_\nu)^{-1}-K)^2 = 0.$$

(iv) For $f \in C$, if $\theta_\nu^2 = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \cdot I(f,K)$, $\sup_{(c_\nu,d_\nu)\in M_\nu} \theta_\nu^2 \le M < \infty$ for all ν.

The linear rank statistics considered are of the form

(2.7) $$S_\nu = \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu) a_\nu(R_{\nu i})$$

with $\gamma_\nu = (\gamma_{\nu 1},\ldots,\gamma_{\nu N_\nu}) \in \mathbb{R}^{N_\nu}$, $\bar{\gamma}_\nu = \sum_{i=1}^{N_\nu} \gamma_{\nu i}/N_\nu$ and $a_\nu(1),\ldots, a_\nu(N_\nu)$ the values of a score function $a_\nu(.)$. The usual regularity condition on the vectors of constants γ_ν is represented by

Condition D.

(i) For $\nu = 1,2,\ldots$, $\sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 > 0$.

(ii) $\lim_{\nu\to\infty} [\sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 / \max_{1\le i\le N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2] = \infty$.

We will say that a sequence of score functions $a_\nu(.)$, $\nu = 1,2,\ldots$, is generated by a real valued function $\phi(u)$, $0 < u < 1$, if

(i) $\int_0^1 \phi^2(u)du < \infty$ and $\int_0^1 (\phi(u)-\bar{\phi})^2 du > 0$ where $\bar{\phi} = \int_0^1 \phi(u)du$.

(ii) $\lim_{\nu\to\infty} \int_0^1 (a_\nu(1+[uN_\nu])-\phi(u))^2 du = 0$ with $[uN_\nu]$ denoting the largest integer not exceeding uN_ν.

In Hájek and Šidák (1967) (p. 158, 164-165), one can find methods for constructing score functions that are generated by a given function $\phi(u)$.

Further, for an ordered sample $U_\nu^{(1)} < \ldots < U_\nu^{(N_\nu)}$ from the uniform distribution on $[0,1]$, we will let

(2.8) $$a_\nu(i,f) = E\phi(U_\nu^{(i)},f) \text{ and } a_{1\nu}(i,f) = E\phi_1(U_\nu^{(i)},f),$$

$i = 1, \ldots, N_\nu$; then, one can easily show that if $f \in C$ and $K \in \mathbb{R}$, the sequence of score functions

$$(2.9) \qquad a_\nu(.,f,K) = Ka_\nu(.,f) + a_{1\nu}(.,f),$$

$\nu = 1,2,\ldots$, is generated by $\phi(u,f,K)$, $0 < u < 1$.

Finally, $\Phi(.)$ will denote the standardized normal distribution function and $k_{1-\alpha}$, the $(1-\alpha)$-quantile of the standardized normal distribution. By convention, for $\sigma^2 = 0$, we will let

$$(2.10) \qquad \Phi(x/\sigma) = \begin{cases} 1 \text{ if } x \geq 0, \\ 0 \text{ if } x < 0. \end{cases}$$

3. Asymptotic distribution under contiguous alternatives

Under H_ν, it is well known from Hâjek (1962) or Hâjek and Šidâk (1967), p. 163, that if condition D is satisfied and $a_\nu(.)$, $\nu = 1,2,\ldots$, are generated by a function $\phi(u)$, $0 < u < 1$, then, the statistics S_ν given by (2.7) are asymptotically normal $(0,\sigma_\nu^2)$ with

$$\sigma_\nu^2 = \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 \cdot \int_0^1 (\phi(u)-\bar{\phi})^2 du. \tag{3.1}$$

For the alternatives K_ν defined by (2.1), the following results will be proved in section 6.

Theorem 3.1. Suppose that a sequence of vectors c_ν and d_ν satisfies condition B. Then, K_ν are contiguous to H_ν.

Theorem 3.2. If $a_\nu(.)$, $\nu = 1,2,\ldots$, are generated by a function $\phi(u)$, $0 < u < 1$, if conditions D and A are satisfied and if for $\nu = 1,2,\ldots$, $\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)^2 \le b^2 (0 \le b^2 < \infty)$ then under K_ν, the statistics S_ν given by (2.7) are asymptotically normal (μ_ν,σ_ν^2) with

$$\mu_\nu = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) \cdot \int_0^1 \phi(u)\phi(u,f,K)du \tag{3.2}$$

and σ_ν^2 given by (3.1).

Beran (1970) has found the asymptotic distribution of linear rank statistics under contiguous alternatives indexed by a q-dimensional parameter. Although his results are more general, the conditions under which they hold are non comparable with the conditions obtained here for the special case of the location and scale parameters. For example, if N_ν is a multiple of 4 ($\nu=1,2,\ldots$) and we define

$$c_{\nu i} = \begin{cases} 0 \text{ if } 1 \le i \le N_\nu/2 \\ (N_\nu)^{-\frac{1}{2}} \text{ if } N_\nu/2 < i \le 3N_\nu/4 \\ -(N_\nu)^{-\frac{1}{2}} \text{ if } 3N_\nu/4 < i \le N_\nu \end{cases} \tag{3.3}$$

and,

$$(3.4) \qquad d_{\nu i} = \begin{cases} -(N_\nu)^{-\frac{1}{2}} \text{ if } 1 \leq i \leq N_\nu/2 \\ (N_\nu)^{-\frac{1}{2}} \text{ if } N_\nu/2 < i \leq 3N_\nu/4 \\ 0 \text{ if } 3N_\nu/4 < i \leq N_\nu \end{cases}$$

($\nu=1,2,\ldots$), one can easily verify that condition (3.20) of Beran is satisfied while our condition A is not. On the other hand, the double-exponential density function belongs to our class C but it fails to satisfy Beran's condition A.

4. Asymptotic sufficiency and asymptotic optimality

The definition of asymptotically sufficient for distinguishing between H_ν and K_ν, given by Hájek and Šidák (1967), p.243-245, can be reformulated for the problem considered here in the following way.

Definition 4.1. The vectors of ranks $R_\nu = (R_{\nu 1},\ldots,R_{\nu N_\nu})$ is asymptotically sufficient for distinguishing between H_ν and

(4.1) $$K_\nu = \{q_\nu : (c_\nu,d_\nu) \in M_\nu\}$$

where q_ν is given by (2.1) and M_ν is a subset of $\mathbb{R}^{N_\nu} \times \mathbb{R}^{N_\nu}$, if

(i) there are densities $p_\nu = p_\nu(x_1,\ldots,x_{N_\nu};\bar{c}_\nu,\bar{d}_\nu) \in H_\nu$ and rank statistics $h_\nu = h_\nu(r_{\nu 1},\ldots,r_{\nu N_\nu};c_\nu^0,d_\nu^0)$ such that for $(c_\nu,d_\nu) \in M_\nu$, the functions

$$q_\nu^0 = p_\nu \cdot h_\nu$$

are densities (ν=1,2,...).

(ii) $\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu) \in M_\nu} \| q_\nu - q_\nu^0 \| = 0$ where $\| p-q \|$ denotes the L_1-distance of two probability densities:

$$\| p-q \| = \int |p-q| d\mu$$

with μ being a σ-finite measure with respect to which the densities are defined.

The following results will be proved in section 7.

Theorem 4.1. If the sequence M_ν satisfies condition M, the vector of ranks R_ν is asymptotically sufficient for distinguishing between H_ν and K_ν where K_ν is given by (4.1).

Theorem 4.2. Consider testing H_ν versus K_ν given by (4.1) and, assume that the sequence M_ν satisfies condition M. Denote by $\beta(\alpha,H_\nu,K_\nu)$ the power of the maximin most powerful test, and by $\bar{\beta}(\alpha,H_\nu,K_\nu)$ the power of the maximin most powerful rank test. Then,

$$\lim_{\nu\to\infty} [\beta(\alpha,H_\nu,K_\nu) - \bar{\beta}(\alpha,H_\nu,K_\nu)] = 0, \; 0 \leq \alpha \leq 1. \tag{4.2}$$

From theorem 4.2, the asymptotically maximin most powerful test for H_ν versus K_ν can be found among the tests based on ranks. The theorem, however, does not specify this test. For the special case where for $\nu = 1,2,\ldots,$ the subset M_ν contains a unique pair of vectors (c_ν,d_ν), the following theorem 4.3 provides an alternate proof of the result of theorem 4.2 and specifies the asymptotically most powerful test explicitly.

Theorem 4.3. Suppose that the sequences of vectors c_ν and d_ν satisfy condition B. Then, the test based on

$$S_\nu^0 = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)a_\nu(R_{\nu i},f,K) \tag{4.3}$$

with critical region $S_\nu^0 \geq k_{1-\alpha}b$ is an asymptotically most powerful test for H_ν versus q_ν at level α. Furthermore, the asymptotic power is given by $1 - \Phi(k_{1-\alpha}-b)$.

Corollary 4.1. The results of theorem 4.3 still hold if the score functions $a_\nu(.,f,K)$ are replaced by score functions $a_\nu(.)$ generated by $\phi(u,f,K)$, $0 < u < 1$.

Corollary 4.2. In theorem 4.3 and corollary 4.1, the densities q_ν can be replaced by

$$q_{\nu,\omega} = \prod_{i=1}^{N_\nu} e^{-(c_{\nu i}+\omega)} f(e^{-(c_{\nu i}+\omega)} x_i - d_{\nu i}) \tag{4.4}$$

where $\omega \in \mathbb{R}$ is unknown and, the test based on S_ν^0 is then an asymptotically uniformly most powerful test for H_ν versus $\{q_{\nu,\omega} : \omega \in \mathbb{R}\}$ at level α.

If we let $d_{\nu i} = 0$ ($i=1,\ldots,N_\nu$ and $\nu=1,2,\ldots$) in theorem 4.3, we obtain the solution of Hájek and Šidák (1967), p.250-251, for scale alternatives. Their solution for location alternatives can also be obtained by transposing the expressions of sections 3 and 4 in terms of $(d_{\nu i}-\bar{d}_\nu)$ instead of $(c_{\nu i}-\bar{c}_\nu)$ and then, setting $c_{\nu i} = 0$ ($i=1,\ldots,N_\nu$ and $\nu=1,2,\ldots$).

5. Two-sample case

Let (m_ν, n_ν), $\nu = 1,2,\ldots$, be a sequence of pairs of positive integers such that $N_\nu = m_\nu + n_\nu \to \infty$ when $\nu \to \infty$. For each ν, define

$$c_{\nu i} = \begin{cases} \Delta_1 (m_\nu n_\nu / N_\nu)^{-1/2} & \text{if } i = 1,\ldots, m_\nu \\ 0 \text{ if } i = m_\nu+1,\ldots, N_\nu \end{cases} \tag{5.1}$$

$$d_{\nu i} = \begin{cases} \Delta_2 (m_\nu n_\nu / N_\nu)^{-1/2} & \text{if } i = 1,\ldots, m_\nu \\ 0 \text{ if } i = m_\nu+1,\ldots, N_\nu \end{cases}$$

where $\Delta = (\Delta_1, \Delta_2) \in \mathbb{R}^2$. The density (2.1) can now be rewritten as

$$q_{\nu,\Delta} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1 (m_\nu n_\nu / N_\nu)^{-1/2}) f(\exp(-\Delta_1 (m_\nu n_\nu / N_\nu)^{-1/2}) x_i - \Delta_2 (m_\nu n_\nu / N_\nu)^{-1/2}) \prod_{i=m_\nu+1}^{N_\nu} f(x_i) \tag{5.2}$$

where f is a density function in C. In the following theorem, the asymptotic distribution, under $q_{\nu,\Delta}$, of statistics of the form (2.7) is given.

Theorem 5.1. Let $a_\nu(.)$, $\nu = 1,2,\ldots$, be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and $\gamma_{\nu i} = 1$ if $i = 1,\ldots, m_\nu$ or, $= 0$ if $i = m_\nu+1,\ldots, N_\nu$ $(\nu=1,2,\ldots)$. Then, if $\Delta_1 \neq 0$ and $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$, the statistics $(m_\nu n_\nu / N_\nu)^{-1/2} S_\nu$ where S_ν is given by (2.7) are, under $q_{\nu,\Delta}$, asymptotically normal with mean

$$\int_0^1 \phi(u)(\Delta_2 \phi(u,f) + \Delta_1 \phi_1(u,f))du \tag{5.3}$$

and variance

$$\int_0^1 (\phi(u) - \bar{\phi})^2 du. \tag{5.4}$$

The asymptotically optimum tests for H_ν versus $q_{\nu,\Delta}$ are given in the following theorems.

Theorem 5.2. Suppose that $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$. Then, the test based on

$$(5.5) \qquad S_{\nu,\Delta} = \sum_{i=1}^{m_\nu} a_\nu(R_{\nu i}, f, \Delta_2/\Delta_1)$$

with critical region

$$(5.6) \qquad (m_\nu n_\nu/N_\nu)^{-1/2}(\Delta_1/|\Delta_1|)S_{\nu,\Delta} \geqq k_{1-\alpha} I^{1/2}(f, \Delta_2/\Delta_1)$$

is an asymptotically most powerful test for H_ν versus $q_{\nu,\Delta}$ where $\Delta_1 \neq 0$, at level α. Furthermore, the asymptotic power is given by $1 - \Phi(k_{1-\alpha} - |\Delta_1| I^{1/2}(f, \Delta_2/\Delta_1))$.

Theorem 5.3. Suppose that $\min(m_\nu, n_\nu) \to \infty$ when $\nu \to \infty$ and let

$$(5.7) \qquad S'_{\nu,\Delta} = \sum_{i=1}^{m_\nu} a_\nu(R_{\nu i}, f, \ell).$$

The test based on $S'_{\nu,\Delta}$ with critical region

$$(5.8) \qquad (m_\nu n_\nu/N_\nu)^{-1/2}\, S'_{\nu,\Delta} \geq k_{1-\alpha} I^{1/2}(f, \ell)$$

is an asymptotically uniformly most powerful α level test for H_ν versus $\{q_{\nu,\Delta} : \Delta_1 > 0,\ \Delta_2/\Delta_1 = \ell\}$.

The test based on $S'_{\nu,\Delta}$ with critical region

$$(5.9) \qquad (m_\nu n_\nu/N_\nu)^{-1/2} S'_{\nu,\Delta} \leq k_\alpha I^{1/2}(f, \ell)$$

is an asymptotically uniformly most powerful α level test for H_ν versus $\{q_{\nu,\Delta} : \Delta_1 < 0,\ \Delta_2/\Delta_1 = \ell\}$.

Corollary 5.1. In theorems 5.2 and 5.3, the densities $q_{\nu,\Delta}$ can be replaced by

$$(5.10)\qquad q'_{\nu,\Delta} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}) f(\exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}})(x_i-\Delta_2(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}})) \prod_{i=m_\nu+1}^{N_\nu} f(x_i).$$

Corollary 5.2. In theorems 5.2 and 5.3, if the densities $q_{\nu,\Delta}$ are replaced by

$$(5.11)\qquad q_{\nu,\Delta,\omega} = \prod_{i=1}^{m_\nu} \exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}-\omega) f(\exp(-\Delta_1(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}-\omega) x_i-\Delta_2(m_\nu n_\nu/N_\nu)^{-\frac{1}{2}}) \prod_{i=m_\nu+1}^{N_\nu} e^{-\omega} f(e^{-\omega} x_i)$$

where $\omega \in \mathbb{R}$ is unknown, then the test based on $S_{\nu,\Delta}$ with critical region given by (5.6) is an asymptotically uniformly most powerful α level test for H_ν versus $\{q_{\nu,\Delta,\omega} : \Delta_1 \neq 0,\ \omega \in \mathbb{R}\}$, the test based on $S'_{\nu,\Delta}$ with critical region given by (5.8) is an asymptotically uniformly most powerful α level test for H_ν versus $\{q_{\nu,\Delta,\omega} : \Delta_1 > 0,\ \Delta_2/\Delta_1 = \ell,\ \omega \in \mathbb{R}\}$ and the test based on $S'_{\nu,\Delta}$ with critical region given by (5.9) is an asymptotically uniformly most powerful α level test for H_ν versus $\{q_{\nu,\Delta,\omega} : \Delta_1 < 0,\ \Delta_2/\Delta_1 = \ell,\ \omega \in \mathbb{R}\}$.

Corollary 5.3. The results of theorems 5.2, 5.3 and corollaries 5.1, 5.2 still hold if the score functions $a\,(.,f,\ell)$ are replaced by score functions $a_\nu(.)$ generated by $\phi(u,f,\ell)$, $0 < u < 1$.

6. Proof of the results of section 3

Define for $i = 1,\ldots, N_\nu$ and $\nu = 1,2,\ldots$ the real functions

$$k_{\nu i}(x) = \frac{\exp(-1/2(c_{\nu i}-\bar{c}_\nu))s(\exp(-c_{\nu i}+\bar{c}_\nu)-e_{\nu i}) - s(x-e_{\nu i})}{c_{\nu i} - \bar{c}_\nu},$$

$$(6.1) \qquad l_{\nu i}(x) = \frac{s(x-e_{\nu i}) - s(x)}{c_{\nu i} - \bar{c}_\nu},$$

$$h_{\nu i}(x) = k_{\nu i}(x) + l_{\nu i}(x)$$

with $s(x) = [f(x)]^{1/2}$ where $f(x)$ is a density function in C. For the proof of theorem 3.1, the following lemmas are needed.

Lemma 6.1. Suppose that the sequences of vectors c_ν and d_ν satisfy condition A. Then,

$$\lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} \int_{-\infty}^{\infty} (h_{\nu i}(x) + \tfrac{1}{2}s(x) + (x+K)s'(x))^2 dx = 0.$$

Proof. Observe first that $\max_{1\leq i\leq N_\nu} \int_{-\infty}^{\infty} h^2_{\nu i}(x)dx < \infty$, $\nu = 1,2,\ldots$, and

$$(6.2) \qquad I(f,K) = 4\int_{-\infty}^{\infty} (-\tfrac{1}{2}s(x) - (x+K)s'(x))^2 dx < \infty.$$

Also, since $s(x)$ is absolutely continuous, we have for almost all x

$$(6.3) \qquad \lim_{\substack{h_1\to 0\\ h_2\to 0}} s(e^{h_1}x+h_2) = s(x) \quad \text{and} \quad \lim_{y\to x} \frac{s(y) - s(x)}{y - x} = s'(x).$$

From condition A and (6.3), we deduce that for almost all x

$$(6.4) \qquad \begin{aligned} &\lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} k_{\nu i}(x) = -\tfrac{1}{2}s(x) - xs'(x),\\ &\lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} l_{\nu i}(x) = -Ks'(x),\\ &\lim_{\nu\to\infty} \max_{1\leq i\leq N_\nu} h_{\nu i}(x) = -\tfrac{1}{2}s(x) - (x+K)s'(x). \end{aligned}$$

Furthermore, by the Cauchy-Schwarz inequality, we have

$$k_{\nu i}^2(x) = \left[\frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}\left(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3t}{2}}xs'(e^{-t}x-e_{\nu i})\right)dt\right]^2$$

(6.5)

$$\leq \frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}\left(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3t}{2}}xs'(e^{-t}x-e_{\nu i})\right)^2dt$$

and,

$$l_{\nu i}^2(x) = \left[\frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{e_{\nu i}}(-s'(x-t))dt\right]^2$$

(6.6)

$$\leq \frac{e_{\nu i}}{c_{\nu i}-\bar{c}_\nu}\int_0^{e_{\nu i}}(-s'(x-t))^2dt$$

so that by Tonelli's theorem

$$\int_{-\infty}^{\infty}k_{\nu i}^2(x)dx \leq \frac{1}{c_{\nu i}-\bar{c}_\nu}\int_0^{c_{\nu i}-\bar{c}_\nu}\int_{-\infty}^{\infty}\left(-\tfrac{1}{2}e^{-\frac{1}{2}t}s(e^{-t}x-e_{\nu i})-e^{-\frac{3}{2}t}xs'(e^{-t}x-e_{\nu i})\right)^2dxdt$$

(6.7)

$$= \int_{-\infty}^{\infty}\left(-\tfrac{1}{2}s(x)-(x+e_{\nu i})s'(x)\right)^2dx$$

and,

$$\int_{-\infty}^{\infty}l_{\nu i}^2(x)dx \leq \frac{e_{\nu i}}{(c_{\nu i}-\bar{c}_\nu)^2}\int_0^{e_{\nu i}}\int_{-\infty}^{\infty}(-s'(x-t))^2dxdt$$

(6.8)

$$= \frac{e_{\nu i}^2}{(c_{\nu i}-\bar{c}_\nu)^2}\int_{-\infty}^{\infty}(-s'(x))^2dx.$$

We can thus conclude from (6.4), (6.7) and (6.8) by means of theorems II.4.2 and V.1.3 of Hájek and Šidák (1967) that

(6.9) $$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} (k_{\nu i}(x) + \tfrac{1}{2}s(x) + xs'(x))^2 dx = 0$$

and

(6.10) $$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \int_{-\infty}^{\infty} (l_{\nu i}(x) + Ks'(x))^2 dx = 0.$$

Consequently, the result follows. □

For a density function $f \in C$ and a sequence of vectors c_ν and d_ν satisfying condition A, define for $\nu = 1,2,\ldots$ the statistics

(6.11) $$T_\nu = -\sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)[1 + (e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu+K)\frac{f'(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}{f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}],$$

(6.12) $$J_\nu = 2\sum_{i=1}^{N_\nu} [(\frac{e^{-c_{\nu i}}f(e^{-c_{\nu i}}X_{\nu i}-d_{\nu i})}{e^{-\bar{c}_\nu}f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)})^{\frac{1}{2}} - 1],$$

and

(6.13) $$L_\nu = \prod_{i=1}^{N_\nu} L_{\nu i}$$

where for $i = 1,\ldots, N_\nu$

(6.14) $$L_{\nu i} = \frac{e^{-c_{\nu i}}f(e^{-c_{\nu i}}X_{\nu i}-d_{\nu i})}{e^{-\bar{c}_\nu}f(e^{-\bar{c}_\nu}X_{\nu i}-\bar{d}_\nu)}.$$

Lemma 6.2. Suppose that the sequences of vectors c_ν and d_ν satisfy condition B. Then, we have

$$\lim_{\nu\to\infty} E(J_\nu) = -\tfrac{1}{4}b^2 \quad\text{and}\quad \lim_{\nu\to\infty} Var(J_\nu-T_\nu) = 0$$

under $\bar{P}_\nu$ where $\bar{P}_\nu$ is the probability measure corresponding to the density

$$(6.15)\qquad \bar{p}_\nu = \prod_{i=1}^{N_\nu} e^{-\bar{c}_\nu} f(e^{-\bar{c}_\nu} x_i - \bar{d}_\nu).$$

Proof. Obviously

$$(6.16)\qquad E(J_\nu) = -\sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \int_{-\infty}^{\infty} h_{\nu i}^2(x)dx$$

and,

$$(6.17)\qquad \mathrm{Var}(J_\nu - T_\nu) \le E(J_\nu - T_\nu)^2$$

$$= 4 \sum_{i=1}^{N_\nu} (c_{\nu i} - \bar{c}_\nu)^2 \int_{-\infty}^{\infty} (h_{\nu i}(x) + \tfrac{1}{2}s(x) + (x+K)s'(x))^2 dx.$$

Thus, by lemma 6.1 and part (ii) of condition B, the lemma is established. □

Lemma 6.3. Suppose that the sequences of vectors c_ν and d_ν satisfy condition A. Then, for arbitrary $\varepsilon > 0$,

$$\lim_{\nu\to\infty} \max_{1\le i\le N_\nu} \bar{P}_\nu(|L_{\nu i} - 1| > \varepsilon) = 0$$

where $\bar{P}_\nu$ is given by (6.15).

Proof. We have by part (i) of condition A and lemma 6.1 that under $\bar{P}_\nu$,

$$(6.18)\qquad \lim_{\nu\to\infty} \max_{1\le i\le N_\nu} E(\sqrt{L_{\nu i}} - 1)^2 = 0.$$

Thus, by the Markov inequality and corollary 5.1.2 of Billingsley (1968), the lemma is established. □

Proof of theorem 3.1. From lemma 6.2 and since that under $\bar{P}_\nu$

$$(6.19)\qquad E(T_\nu) = 0 \quad \text{and} \quad \lim_{\nu\to\infty} \mathrm{Var}(T_\nu) = b^2,$$

it follows that under $\bar{P}_\nu$

(6.20) $$\lim_{\nu\to\infty} E(J_\nu - T_\nu + \tfrac{1}{4}b^2)^2 = 0.$$

By theorem V.1.2 of Hájek and Šidák (1967) we have T_ν asymptotically normal $(0,b^2)$ under $\bar{P}_\nu$ and by (6.20) we have then that J_ν are asymptotically normal $(-\tfrac{1}{4}b^2,b^2)$ under $\bar{P}_\nu$. This entails with lemma 6.3 and Le Cam's second lemma (see Hájek and Šidák (1967), p.205) that

(6.21) $$\lim_{\nu\to\infty} \bar{P}_\nu(|\ln L_\nu - J_\nu + \tfrac{1}{2}b^2| > \varepsilon) = 0$$

for arbitrary $\varepsilon > 0$ and, $\ln L_\nu$ asymptotically normal $(-\tfrac{1}{2}b^2,b^2)$ under $\bar{P}_\nu$. Consequently, since $\bar{P}_\nu \in H_\nu$, the corollary of Le Cam's first lemma (see Hájek and Šidák (1967), p.204) completes the proof. □

For $i = 1,\ldots, N_\nu$ and $\nu = 1,2,\ldots$, we introduce the random variables

(6.22) $$U_{\nu i} = F(e^{-\bar{c}_\nu} X_{\nu i} - \bar{d}_\nu)$$

where F is the distribution function of a density $f \in C$. Under $\bar{P}_\nu$, the random variables $U_{\nu 1},\ldots, U_{\nu N_\nu}$ are independently uniformly distributed on [0,1]. The next two lemmas are needed in the proof of theorem 3.2.

Lemma 6.4. Let $a_\nu(.)$, $\nu = 1,2,\ldots$, be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and assume that the sequence of vectors γ_ν satisfies condition D. Then, for S_ν given by (2.7) and

(6.23) $$T_\nu^\phi = \sum_{i=1}^{N_\nu} (\gamma_{\nu i} - \bar{\gamma}_\nu)\phi(U_{\nu i}),$$

we have for arbitrary $\varepsilon > 0$

$$\lim_{\nu\to\infty} \bar{P}_\nu(|S_\nu - T_\nu^\phi| > \varepsilon) = 0$$

where $\bar{P}_\nu$ is given by (6.15).

The proof of this lemma is similar to the arguments of Hājek and Šidāk (1967), p.160-161 and 164-165.

Lemma 6.5. Let $a_\nu(.)$, $\nu = 1,2,\ldots$, be a sequence of score functions generated by a function $\phi(u)$, $0 < u < 1$, and suppose that the sequences of vectors c_ν and d_ν satisfy condition B. Assume also that the sequence of vectors γ_ν satisfies condition D and,

$$(6.24) \qquad \lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) = b_{12}.$$

Then, for $\bar{P}_\nu$, T_ν^ϕ and T_ν given respectively by (6.15), (6.23) and (6.11), we have that under $\bar{P}_\nu$, (T_ν^ϕ, T_ν) are asymptotically jointly normal with mean vector (0,0) and covariance matrix $\begin{pmatrix} \sigma^2 & \sigma_{12} \\ \sigma_{12} & b^2 \end{pmatrix}$ where

$$(6.25) \qquad \sigma^2 = \int_0^1 (\phi(u)-\bar{\phi})^2 du$$

and,

$$(6.26) \qquad \sigma_{12} = b_{12} \int_0^1 \phi(u)\phi(u,f,K)du.$$

Proof. Since from (6.11) and (6.22), we can write

$$(6.27) \qquad T_\nu = \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)\phi(U_{\nu i},f,K),$$

the proof of this lemma is obtained by arguments similar to Hājek and Šidāk (1967), p.217-218. □

Proof of theorem 3.2. Without loss of generality one can suppose that

$$(6.28) \qquad \sum_{i=1}^{N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 = 1, \quad \nu = 1,2,\ldots .$$

Then, from condition D, it follows that

$$(6.29) \qquad \lim_{\nu\to\infty} \max_{1\le i\le N_\nu} (\gamma_{\nu i}-\bar{\gamma}_\nu)^2 = 0.$$

It is sufficient to prove the theorem under the additional assumptions:

(6.30) $$\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu)(\gamma_{\nu i}-\bar{\gamma}_\nu) = b_{12}$$

and,

(6.31) $$\lim_{\nu\to\infty} \sum_{i=1}^{N_\nu} (c_{\nu i}-\bar{c}_\nu).I(f,K) = b_1^2 \text{ with } 0 \le b_1^2 < \infty.$$

Indeed, if the theorem were false, there would exists a subsequence of $\{\nu\}$ with the property that for all its subsequences the theorem would fail to hold. However, every subsequence has a further subsequence such that (6.30) and (6.31) hold. That the theorem is true under the assumptions (6.28), (6.29), (6.30) and (6.31) can be seen as follows.

Suppose first $b_1^2 > 0$. From (6.20), (6.21) and lemma 6.4, we have that under $\bar{P}_\nu$, $(S_\nu, \ln L_\nu)$ has the same asymptotic distribution as $(T_\nu^\phi, T_\nu - \frac{1}{2}b_1^2)$. Thus, from lemma 6.5, it follows that under $\bar{P}_\nu$, $(S_\nu, \ln L_\nu)$ is asymptotically jointly normal with mean vector (0,0) and covariance matrix $\begin{pmatrix} \sigma^2 & \sigma_{12} \\ \sigma_{12} & b_1^2 \end{pmatrix}$ where σ^2 and σ_{12} are respectively given by (6.25) and (6.26). By Le Cam's third lemma (see Hájek and Šidák (1967), p.208), we conclude that S_ν are asymptotically normal (σ_{12}, σ^2) under K_ν.

The case $b_1^2 = 0$ follows from the remarks of Hájek and Šidák (1967), p.210 and 219. This completes the proof. □

7. Proof of the results of section 4

Before presenting the proof of Theorem 4.1, it is usefull to give the next two lemmas.

Lemma 7.1. If the sequence M_ν satisfies condition M, then, for $\bar{p}_\nu$ and S_ν^0 given by respectively (6.15) and (4.3), we have

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \bar{P}_\nu(|q_\nu/\bar{p}_\nu-\exp(S_\nu^0-\tfrac{1}{2}\theta_\nu^2)| > \varepsilon) = 0 \tag{7.1}$$

for arbitrary $\varepsilon > 0$.

Proof. We shall first show that (7.1) is implied by

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \bar{P}_\nu(|\ln(q_\nu/\bar{p}_\nu)-S_\nu^0+\tfrac{1}{2}\theta_\nu^2| > \varepsilon) = 0 \tag{7.2}$$

for arbitrary $\varepsilon > 0$.

Since $E(q_\nu/\bar{p}_\nu) \leq 1$, the Markov inequality gives us that for every $\eta > 0$, there exist $\delta = \delta(\eta) > 0$ such that

$$\bar{P}_\nu(q_\nu/\bar{p}_\nu \geq \delta) \leq \eta. \tag{7.3}$$

Let $\alpha = \ln(1+\varepsilon/\delta(\eta))$ with $\varepsilon > 0$. From

$$\begin{aligned}&\bar{P}_\nu(\{|\ln(q_\nu/\bar{p}_\nu)-S_\nu^0+\tfrac{1}{2}\theta_\nu^2| < \alpha\} \cap \{q_\nu/\bar{p}_\nu \leq \delta(\eta)\})\\ &\qquad \leq \bar{P}_\nu(|q_\nu/\bar{p}_\nu-\exp(S_\nu^0-\tfrac{1}{2}\theta_\nu^2)| \leq \varepsilon)\end{aligned} \tag{7.4}$$

we may conclude that for every $\varepsilon > 0$ and $\eta > 0$,

$$\begin{aligned}&\bar{P}_\nu(|\ln(q_\nu/\bar{p}_\nu)-S_\nu^0+\tfrac{1}{2}\theta_\nu^2| < \alpha)\\ &\qquad \leq \bar{P}_\nu(|q_\nu/\bar{p}_\nu-\exp(S_\nu^0-\tfrac{1}{2}\theta_\nu^2)| \leq \varepsilon) + \eta.\end{aligned} \tag{7.5}$$

Then, by taking $\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu}$ on each side of (7.5) and noting (7.2), we get (7.1).

Now, (7.1) may be proved by reasoning similar to Hájek and Šidák (1967), p.246, i.e. by assuming that (7.1) is false thus, that (7.2) is false and then drawing a contradictory subsequence by making use of (6.20), (6.21) and the fact that $T_\nu - S_\nu^0 \to 0$ in $\bar{P}_\nu$-probability as $\nu \to \infty$, which can be proved as in Hájek and Šidák (1967), p.161. □

Lemma 7.2. Suppose that the sequence M_ν, $\nu = 1,2,\ldots$, satisfies condition M and let $\{\nu_k\}$ be a strictly increasing subsequence of $\{\nu\}$. Then, for the sequence C_ν define by $C_\nu = k$ if $\nu_k \leq \nu < \nu_{k+1}$, we have

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \left| 1 - \int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta_\nu^2)d\bar{P}_\nu(S_\nu^0 \leq x)\right| = 0$$

where $\bar{P}_\nu$ and S_ν^0 are respectively given by (6.15) and (4.3).

Proof. From part (iv) of condition M, we may deduce

$$\lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \left| \int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta_\nu^2)d\Phi(x/\theta_\nu) - 1\right|$$

(7.6)

$$\leq \lim_{\nu\to\infty} \sup_{0\leq\theta_\nu^2\leq M} \left|\Phi((C_\nu-M)/\theta_\nu)-\Phi((-C_\nu-M)/\theta_\nu) - 1\right| = 0.$$

Assume now the existence of an $\varepsilon_0 > 0$ and a subsequence $\{\nu_j\} \subset \{\nu_k\}$ such that

$$(7.7)\qquad \left|\int_{-C_{\nu_j}}^{C_{\nu_j}} \exp(x-\tfrac{1}{2}\theta_{\nu_j}^2)d\bar{P}_{\nu_j}(S_{\nu_j}^0 \leq x) - \int_{-C_{\nu_j}}^{C_{\nu_j}} \exp(x-\tfrac{1}{2}\theta_{\nu_j}^2)d\Phi(x/\theta_{\nu_j})\right| > \varepsilon_0.$$

But, since the sequence M_{ν_j}, $j = 1,2,\ldots$, satisfies condition M, the sequence $\{\nu_j\}$ contains a subsequence $\{\nu_\ell\}$ such that

$$(7.8)\qquad \lim_{\ell\to\infty} \theta_{\nu_\ell}^2 = b^2,\ 0 \leq b^2 < \infty.$$

And, from Hájek and Šidák (1967), p.163, we have that under $\bar{P}_{\nu_\ell}$ the statistics

$S^0_{\nu_\ell}$ are asymptotically normal $(0,\theta^2_{\nu_\ell})$. Let

$$(7.9)\qquad h^0_{\nu_\ell}(x) = \begin{cases} \exp(x-\tfrac{1}{2}\theta^2_{\nu_\ell}) \text{ if } |x| \le C_{\nu_\ell}, \\ 0 \text{ if } |x| > C_{\nu_\ell} \end{cases}$$

and denote by E the set of x such that $h^0_{\nu_\ell}(x_\ell) \to \exp(x-\frac{1}{2}b^2)$ for some sequence x_ℓ approaching x. Since the complement of E is empty and the random variables $h^0_{\nu_\ell}(S^0_{\nu_\ell})$ are uniformly integrable, we conclude from Billingsley (1968), p.32-34, that

$$(7.10)\qquad \lim_{\ell\to\infty} \left| \int_{-C_{\nu_\ell}}^{C_{\nu_\ell}} \exp(x-\tfrac{1}{2}\theta^2_{\nu_\ell})\, d\bar{P}_{\nu_\ell}(S^0_{\nu_\ell} \le x) - 1 \right| = 0.$$

Thus, by combining (7.10) with (7.6) we contradict (7.7). The proof is finished. □

Proof of theorem 4.1. Let $p_\nu = \bar{p}_\nu$ where $\bar{p}_\nu$ is given by (6.15) and, define

$$(7.11)\qquad h_\nu = \begin{cases} B_\nu \exp(S^0_\nu-\tfrac{1}{2}\theta^2_\nu) \text{ if } |S^0_\nu| \le C_\nu, \\ 0 \text{ if } |S^0_\nu| > C_\nu \end{cases}$$

where

$$(7.12)\qquad B_\nu = \left[\int_{-C_\nu}^{C_\nu} \exp(x-\tfrac{1}{2}\theta^2_\nu) dP_\nu(S^0_\nu \le x)\right]^{-1},$$

S^0_ν is given by (4.3), P_ν is the probability measure corresponding to p_ν and C_ν, $\nu = 1,2,\ldots$, is a sequence of reals such that $C_\nu > 0$ and $C_\nu \to \infty$ when $\nu \to \infty$. This sequence will be specified later.

Obviously, $p_\nu \in H_\nu$ and h_ν is a rank statistics depending on the vectors c^0_ν and d^0_ν only since S^0_ν as the same property. Furthermore, the functions $q^0_\nu = p_\nu . h_\nu$ provide densities for $(c_\nu, d_\nu) \in M_\nu$ $(\nu=1,2,\ldots)$. Consequently, it remains to show that

$$(7.13)\qquad \lim_{\nu\to\infty} \sup_{(c_\nu,d_\nu)\in M_\nu} \| q_\nu - q^0_\nu \| = 0.$$

Since $\| p-q \| = 2 \int_{\{q<p\}} (1-q/p)dP$, we have

$$(7.14) \qquad \| q_\nu - q_\nu^0 \| = 2 \int_{\{q_\nu/p_\nu < h_\nu\}} (h_\nu - \exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2))dP_\nu$$

$$+ 2 \int_{\{q_\nu/p_\nu < h_\nu\}} (\exp(S_\nu - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu)dP_\nu .$$

For each $(c_\nu, d_\nu) \in M_\nu$, the absolute value of the first of the last two integrals is bounded by $|1-B_\nu^{-1}|$ and, the second integral is bounded by $\alpha_\nu \exp(C_\nu) + \varepsilon_\nu$ where ε_ν $(\nu=1,2,\ldots)$ is a sequence of positive real numbers and

$$(7.15) \qquad \alpha_\nu = P_\nu(\exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu > \varepsilon_\nu).$$

Consequently,

$$(7.16) \qquad \| q_\nu - q_\nu^0 \| \le 2 \sup_{(c_\nu, d_\nu)\in M_\nu} |1-B_\nu^{-1}| + 2e^{C_\nu} \sup_{(c_\nu, d_\nu)\in M_\nu} \alpha_\nu + 2\varepsilon_\nu .$$

Let $\varepsilon > 0$ be given. From lemma 7.1, for every integer k, there exists ν_k such that for $\nu > \nu_k$

$$(7.17) \qquad e^k \sup_{(c_\nu, d_\nu)\in M_\nu} P_\nu(\exp(S_\nu^0 - \tfrac{1}{2}\theta_\nu^2) - q_\nu/p_\nu > 1/k) < 1/k.$$

We can assume that the sequence ν_k, $k = 1,2,\ldots$, is strictly increasing and then, define

$$(7.18) \qquad C_\nu = k \text{ for } \nu_k \le \nu < \nu_{k+1}.$$

From lemma 7.2, there exists $\nu_1(\varepsilon)$ such that for $\nu > \nu_1(\varepsilon)$

$$(7.19) \qquad \sup_{(c_\nu, d_\nu)\in M_\nu} |1-B_\nu^{-1}| < \varepsilon/6.$$

For $\varepsilon_\nu = 1/k$ if $\nu_k \le \nu < \nu_{k+1}$, (7.17) becomes

$$(7.20) \qquad e^{C_\nu} \sup_{(c_\nu, d_\nu)\in M_\nu} \alpha_\nu < 1/C_\nu \text{ for } \nu_k \leq \nu < \nu_{k+1},\ k = 1,2,\ldots$$

Thus, there exists $\nu_2(\varepsilon)$ such that for $\nu > \nu_2(\varepsilon)$

$$(7.21) \qquad e^{C_\nu} \sup_{(c_\nu, d_\nu)\in M_\nu} \alpha_\nu < \varepsilon/6 \text{ and } \varepsilon_\nu < \varepsilon/6.$$

From (7.19) and (7.21), (7.13) is deduce and the proof is complete. □

Proof of theorem 4.2. Using theorem 4.1, the result follows from Hājek and Šidāk (1967), p.243-244, with $a = (\bar{c}_\nu, \bar{d}_\nu)$ and $b = (c_\nu^0, d_\nu^0)$. □

Proof of theorem 4.3. In view of theorems 3.1 and 3.2, the proof is similar to Hājek and Šidāk (1967), p.251. □

Proof of corollary 4.1. From theorem 3.2, the asymptotic power of the test is $1 - \Phi(k_{1-\alpha} - b)$ and thus, the result follows. □

Proof of corollary 4.2. Let $c'_{\nu i} = c_{\nu i} + \omega$, $i = 1, \ldots, N_\nu$. The sequence of vectors c'_ν and d_ν satisfies condition B with the same K. Thus, since $c_{\nu i} - \bar{c}_\nu = c'_{\nu i} - \bar{c}'_\nu$, the result is immediate. □

8. Proof of the results of section 5

Proof of theorem 5.1. It may be shown, by easy algebraic transformations, that the sequence of vectors c_ν and d_ν, defined by (5.1), satisfies condition B with

$$K = \Delta_2/\Delta_1 \text{ and } b^2 = \Delta_1^2\, I(f,\Delta_2/\Delta_1). \tag{8.1}$$

Also, from Hâjek and Šidâk (1967), p.162, condition D is verified. Consequently, the result is deduced from theorem 3.2.□

Proof of theorem 5.2. The direct application of (8.1) in theorem 4.3 permits us to conclude the result.□

The other results are deduced from theorem 5.2.

Acknowledgements. I would like to mention my sincere gratitude to Professor Constance van Eeden for suggesting me this problem and also, for her constructive comments during the preparation of the manuscript.

REFERENCES

Beran, R.J. (1970). Linear rank statistics under alternatives indexed by a vector parameter. Ann. Math. Statist. 41 1896-1905.

Billingsley, P. (1968). Convergence of Probability Measures. Wiley, New York.

Hâjek, J. (1962). Asymptotically most powerful rank-order tests. Ann. Math. Statist. 33 1124-1147.

Hâjek, J. and Šidâk, Z. (1967). Theory of Rank Tests. Academic Press, New York.